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A FULLY CONSERVATIVE NUMERICAL SCHEME FOR A KIND OF GENERALIZED 3D PERIODIC EULER EQUATIONS

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ABSTRACT

A kind of generalized 3D Euler equations with periodic boundary conditions, that describes the behaviour of the small scale flow in a turbulence model is solved. A fully conservative numerical scheme, that combines pseudospectral discretization in space with a variation of Crank–Nicholson's scheme in time, is introduced. Some numerical tests show that the numerical solution reaches an asymptotic statistic steady state. In the case of well developed isotropic turbulence, these results are shown to present a reasonable quantitative agreement with the classical theory.

KEY WORDS Crank–Nicholson scheme Pseudospectral methods Vectorial computing Turbulence modelling

INTRODUCTION

This paper deals with the solution of a kind of generalized three-dimensional Euler equations with periodic boundary conditions, that we state below. The numerical solution of this problem appears as a partial question in the asymptotic analysis of viscous flows with highly oscillating initial conditions made by McLaughlin, Papanicolaou and Pironneau[16] and subsequent authors[8,9]. In these works, problem (3) describes the canonical behaviour of the small scale flow. We shall briefly describe the analysis done in these papers in order to give a motivation to our problem and to state it properly.

Let $\varepsilon > 0$ be a small parameter and $T > 0$ a given final time. We shall consider the Navier–Stokes equations for three-dimensional viscous fluids in $\mathbb{R}^3 \times [0, T]$; with kinematic viscosity of order ε^2:

$$\left.\begin{aligned} &u^\varepsilon_{,t} + (u^\varepsilon \cdot \nabla)u^\varepsilon + \nabla p^\varepsilon - \mu\varepsilon^2 \Delta u^\varepsilon = 0, \qquad \nabla \cdot u^\varepsilon = 0 \qquad \text{in } \mathbb{R}^3 \times]0, T[\\ &u^\varepsilon(x, 0) = u_0(x) + \varepsilon^{1/3} w^0\left(\frac{x}{\varepsilon}, x\right) \qquad \text{in } \mathbb{R}^3 \end{aligned}\right\} \tag{1}$$

Here, $u^\varepsilon(x, t)$ and $p^\varepsilon(x, t)$ are the velocity field and the pressure, respectively, and $\mu > 0$ is a given number of order one. Also, u_0 is a smooth velocity field in $\mathbb{R}^3$ and $w^0(y, x)$ is a smooth velocity field in $\mathbb{R}^3 \times \mathbb{R}$. The field $w^0(y, x)$ is supposed to be periodic in the y variable, to be free-divergence in the x and y variables and to have zero mean in the y variable:

$$\langle w^0 \rangle = \langle w^0 \rangle(x) = \frac{\int_Y w^0(y, x)\,\mathrm{d}y}{\int_Y \mathrm{d}y}$$

where Y is a period cell of w^0.

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Note that ε represents the dimensionless ratio of characteristic length scales associated to the initial data $u^\varepsilon(x,0)$. It is known that problem (1), completed with suitable conditions at infinity, has a unique smooth solution if T is small enough[2,12,14,22].

The purpose of the work done in References 8, 9, 16, is to analyse by asymptotic techniques the behaviour of the flow governed by (1) as ε decreases to zero. The velocity field u^ε is assumed to admit an asymptotic expansion of the form:

$$u^\varepsilon(x,t)\sim u(x,t)+\varepsilon^{1/3}w\left(\frac{a(x,t)}{\varepsilon},\frac{t}{\varepsilon^{2/3}};x,t\right)+\varepsilon^{2/3}u^{(1)}\left(\frac{a(x,t)}{\varepsilon},\frac{t}{\varepsilon^{2/3}};x,t\right)+\cdots$$

Here, $u(x,t)$ is the 'mean velocity field', and $w(y,\tau;x,t)$, $u^{(1)}(y,\tau;x,t),\ldots$ are the 'perturbations', that are assumed to be periodic in y, bounded in τ, and to have $y-\tau$ zero mean, defined as:

$$\langle\langle w\rangle\rangle=\langle\langle w\rangle\rangle(x,t)=\lim_{\tau\to\infty}\frac{1}{2\tau}\int_{-\tau}^{\tau}\langle w(\cdot,\sigma;x,t)\rangle\,d\sigma=0$$

Also, $a(x,t)$ are the Lagrangian coordinates associated to the velocity field u:

$$a_{,t}+(u\cdot\nabla)a=0,\quad \text{in } \mathbb{R}^3\times]0,T[;\quad a(x,0)=x \text{ in } \mathbb{R}^3 \tag{2}$$

A formal mathematical derivation is used to obtain equations that determine the terms in the asymptotic expansion. In particular, the perturbation w is determined by means of a 'canonic' Cauchy problem for the following generalized 3D Euler equations in the y and τ variables:

$$\left.\begin{array}{lll}\tilde{w}_{,t}+(\tilde{w}\cdot\nabla_y)\tilde{w}+(C\nabla_y)\pi=0, & \nabla_y\cdot\tilde{w}=0 & \text{in } \mathbb{R}^3\times\mathbb{R}\\ \tilde{w}(y,0)=\tilde{w}_0(y) & & \text{in } \mathbb{R}^3\\ \tilde{w},\pi\quad \text{periodic in } y, \text{bounded in } \tau & & \end{array}\right\} \tag{3}$$

Here, C is a square symmetric matrix of dimension 3×3, depending only on the mean velocity field u, as follows:

$$C=G^{\mathrm{T}}G,\quad \text{with } G=\nabla a,\quad \left(\text{i.e., } G_{ij}=\frac{\partial a_j}{\partial x_i}\right)$$

Note that matrix C depends only on the variables x and t, and acts as a parameter in (3).

A formal mathematical argument fully based on the conservation properties of (3) is used to deduce averaged equations that govern the mean flow of u^ε. These equations involve the mean velocity field u, and also the following statistics of $\tilde{w}$:

- *mean kinetic energy*:

$$q=\frac{1}{2}\langle\tilde{w}\cdot C^{-1}\tilde{w}\rangle$$

- *mean helicity:*

$$h=\frac{1}{2}\langle\tilde{w}\cdot C^{-1}\tilde{r}\rangle$$

where $\tilde{r}=\nabla_y\times(C^{-1}\tilde{w})$.

These averaged equations are the following:

$$\left.\begin{array}{ll}u_{,t}+(u\cdot\nabla)u+\nabla p=0, & \nabla\cdot u=0 \text{ in } \mathbb{R}^3\times[0,T]\\ u(x,0)=u_0(x) & \text{in } \mathbb{R}^3\end{array}\right\} \tag{4}$$

$$\left.\begin{aligned} &q_{,t}+(u\cdot\nabla)q+q\left[(G^{-\mathrm{T}}\tilde{\mathrm{R}}(C)G^{-1})\colon\nabla u+\mu\left(\frac{h}{q}\right)^2\psi_q(C)\right]=0 \text{ in } \mathbb{R}^3\times[0,T] \\ &q(x,0)=q_0(x)=\frac{1}{2}\langle|w^0(\cdot,x)|^2\rangle \qquad \text{in } \mathbb{R}^3 \end{aligned}\right\} \tag{5}$$

$$\left.\begin{aligned} &h_{,t}+(u\cdot\nabla)h+h\left[(G^{-\mathrm{T}}\tilde{\mathrm{S}}(C)G^{-1})\colon\nabla u+\mu\left(\frac{h}{q}\right)^2\psi_h(C)\right]=0 \text{ in } \mathbb{R}^3\times[0,T] \\ &h(x,0)=h_0(x)=\frac{1}{2}\langle w^0(\cdot,x)\cdot(\nabla_y\times\omega^0)(\cdot,x)\rangle \qquad \text{in } \mathbb{R}^3 \end{aligned}\right\} \tag{6}$$

Here, $\tilde{\mathrm{R}}$ and $\tilde{\mathrm{S}}$ are 3×3 tensors, and ψ_q and ψ_h are scalar functions, of the matrix C. All of them are the 'closure terms' of the model. They are given through the solution $\tilde{w}$ of Euler equations (3), as follows:

$$\left.\begin{aligned} &\tilde{\mathrm{R}}(C)=\langle\langle\tilde{w}(C)\otimes\tilde{w}(C)\rangle\rangle; \quad \psi_q(C)=\langle\langle\tilde{r}(C)\cdot C^{-1}\tilde{r}(C)\rangle\rangle \\ &\tilde{\mathrm{S}}(C)=\langle\langle\tilde{r}(C)\otimes\tilde{w}(C)\rangle\rangle; \quad \psi_h(C)=\langle\langle\tilde{r}(C)\cdot C^{-1}(\nabla_y\times C^{-1}\tilde{r}(C))\rangle\rangle \end{aligned}\right\} \tag{7}$$

System (2)–(7) is closed by giving the initial condition $\tilde{w}_0$ of Euler equations (3):

$$\tilde{w}_0(y)=\frac{1}{\sqrt{q_0}}w^0\left(\frac{q_0}{h_0}y\right) \tag{8}$$

Let us remark that once system (2)–(8) is solved, the main perturbation w is determined as follows:

$$w(y,\tau)=\sqrt{q}\,G^{-\mathrm{T}}\tilde{w}\left(\frac{h}{q}y,\frac{q_0}{\sqrt{q}}\tau\right) \tag{9}$$

The set of equations (2)–(8) can be viewed as a model that describes the asymptotic behaviour of u^ε as ε decreases to zero. From the point of view of turbulence modelling, this is a very sophisticated two equations model, whose performances must be analysed in test cases before using it in cases of practical interest.

Some relevant numerical tests show that the model takes into account some qualitative transient effects of the interactions large–small structures, that are misregarded by the usual steady turbulence models[4,6,7,8]. However, little information about the quantitative performances of the model has been obtained up to the present.

The big difficulty in obtaining good quantitative results arises when computing the closure terms of the model. Indeed the generalized Euler equations (3) are solved by means of a steady approach[5,17]. Concretely, the steady equivalent of (3) is solved by a combined least squares–conjugate gradient algorithm. This provides temptative values for the closure terms. However, this least squares formulation provides numerical solutions of (3) that are only locally unique, and that are very sensitive to the initialization. Thus, some of these solutions could not be physically correct. This has proved to happen when solving some steady flow problems, such as transonic flow around obstacles. In this case, some shocks with unphysical locations may appear[11].

Our purpose here is to perform a conservative transient approach to the solution of (3). As the derivation of the model equations is fully based upon the conservation properties of Euler equations, it seems convenient to use numerical schemes that conserve in time τ all quantities involved in the model (mean, mean kinetic energy and mean helicity). Our hope is to compute reasonably accurate closure terms.

First, we introduce two versions of a solver that combines a pseudospectral discretization in space with a variation of Crank–Nicholson's scheme in time. This solver is proved to possess excellent conservation properties. Next we prove that one of the two versions of the solver is second order accurate for smooth enough solutions, under some specific restrictions. The numerical tests performed are then described. A quasi-steady state of the space–time means in increasing finite intervals of the closure terms is found. This indicates that our approach is good to compute the closure terms. Finally, we give some evidence that the computed closure terms are quantitatively correct. Concretely, we verify that the computed values of some meaningful statistics agree with the 'theoretical' values given by the well known theory of isotropic turbulence with constant mean velocity field.

We shall use the following notation for norms and spaces.

Let Y denote the open set defined by:

$$Y =]0, 2\pi[^3$$

Given k integer and $q \in [1, +\infty]$, we shall denote by $W^{k,q}(Y)$ the usual Sovolev function space:

$$W^{k,q}(Y) = \{v\colon Y \to \mathbb{R} \mid D^\alpha v \in L^q(Y),\ \forall \alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3 \text{ such that } \alpha_1 + \alpha_2 + \alpha_3 \leqslant k\}.$$

This is a Banach space endowed with the following norm:

$$\|v\|_{k,q} = \left[\sum_{\alpha_1+\alpha_2+\alpha_3=1}^{k} \|D^\alpha v\|_q^q \right]^{1/q}$$

where $\|\cdot\|_q$ denotes the usual norm in $L^q(Y)$. We shall denote by $W_P^{k,q}(Y)$ the space of functions that belong to $W^{k,q}(\mathbb{R})$ which are periodic with period cell Y.

When $q = 2$ we shall denote $W_P^{k,q}(Y)$ by $H_P^k(Y)$.

Finally, $|\cdot|$ will denote the euclidean norm on $\mathbb{R}^3$.

A FULLY CONSERVATIVE PSEUDOSPECTRAL SOLVER

The discretization in time of flow problems by means of Crank–Nicholson's scheme helps to obtain numerical schemes that reproduce the conservation properties of the continuous equation. This happens, in particular, when solving 2D Euler equations in bounded domains[21], or 3D Navier–Stokes equations with periodic boundary conditions[10]. In our case, we shall use a variation of this scheme to discretize Euler equations (3) in time, together with a pseudospectral discretization in space, in order to obtain a fully conservative solver.

Description and properties of numerical scheme

In order to give a motivation to our actual numerical scheme, let us state some facts concerning (3).

Note at first that the initial perturbation w^0 in (1) is y-periodic with period cell $[0, 2\pi]^3$. Then, the initial condition $\tilde{w}_0$ in (3) is y-periodic with period cell $Y = [0, 2\pi/s]^3$, $s = q_0/h_0$. This makes it necessary to consider discrete periodic functions with such period cell.

Note also that the formula:

$$\tilde{w} \cdot \nabla \tilde{w} = \frac{1}{2}(C\nabla)|G^{-\mathrm{T}}\tilde{w}|^2 - C(\tilde{w} \times \tilde{r}), \quad \text{if } \det(C) = 1 \tag{10}$$

transforms (3) into:

$$\left.\begin{array}{lll} \tilde{w}_{,\tau} - C(\tilde{w} \times \tilde{r} + \nabla p) = 0, & \nabla_y \cdot \tilde{w} = 0 & \text{in } \mathbb{R}^3 \times \mathbb{R} \\ \tilde{w}(y, 0) = \tilde{w}_0(y) & & \text{in } \mathbb{R}^3 \\ \tilde{w},\ p \quad \text{periodic in } y, \text{bounded in } \tau & & \end{array}\right\} \tag{11}$$

where $p=\pi+\frac{1}{2}|G^{-T}\tilde{w}|^2$ and where we have dropped the explicit dependency on the y variable of the space derivatives.

Under this formulation, the conservation of mean, mean kinetic energy and mean helicity in (11) is obtained in a natural way:

$$\langle\tilde{w}\rangle_{,\tau}=0, \quad \langle\tilde{w}\cdot C^{-1}\tilde{w}\rangle_{,\tau}=0, \quad \langle\tilde{w}\cdot C^{-1}\tilde{r}\rangle_{,\tau}=0$$

This indicates the convenience of looking for a numerical scheme that keeps the structure of the equation in (11) if we wish to reproduce these conservation properties.

To describe our scheme, let $m\geqslant 1$ and $s>0$ be given integer and real numbers, respectively. Let us consider the following complex $2\pi/s$-periodic functions:

$$\varphi_\alpha(\xi)=e^{is\xi\alpha}, \quad \forall\xi\in\mathbb{R}, \quad \text{for every } \alpha\in\mathbb{Z}$$

$$\phi_{\mathbf{k}}(y)=e^{is\mathbf{k}\cdot y}, \quad \forall y\in\mathbb{R}^3, \quad \text{for every } \mathbf{k}=(k_1, k_2, k_3)\in\mathbb{Z}^3$$

Let us denote by S_N the space of functions spanned by:

$$\{\varphi_\alpha, \text{ for } \alpha=-N,\ldots,N-1\}$$

and by E_N the space spanned by:

$$\{\phi_{\mathbf{k}}, \text{ for } k_1, k_2, k_3=-N,\ldots,N-1\}$$

Let us denote by $C_p(Y)$ the space of periodic continuous functions on $\mathbb{R}^3$ with period cell $Y=[0, 2\pi/s]^3$, and define the space step $h=2\pi/2Ns$. We shall define implicitly an interpolation operator P_N from $C_p(Y)$ onto $[E_N]^3$ as follows:

$$(P_N v)(\mathbf{j}h)=v(\mathbf{j}h), \quad \text{for } \mathbf{j}=(j_1, j_2, j_3) \quad \text{and} \quad j_1, j_2, j_3=0, 1, \ldots, 2N-1$$

for every $v\in C_p(Y)$.

The set of equalities that appears here constitutes a square linear system whose matrix is proportional to an orthogonal matrix. Its solution is known analytically, and it is given by[15]:

$$P_N v=\sum_{k_1=-N}^{N-1}\sum_{k_2=-N}^{N-1}\sum_{k_3=-N}^{N-1}\hat{v}_{\mathbf{k}}\phi_{\mathbf{k}}$$

with

$$\hat{v}_{\mathbf{k}}=\frac{1}{(2N)^3}\sum_{j_1=0}^{2N-1}\sum_{j_2=0}^{2N-1}\sum_{j_3=0}^{2N-1}v(\mathbf{j}h)\bar{\phi}_{\mathbf{k}}(\mathbf{j}h)$$

Also, we shall understand that if $\tilde{w}=(w_1, w_2, w_3)\in[C_P(Y)]^3$ then

$$P_N\tilde{w}=(P_N w_1, P_N w_2, P_N w_3)\in[E_N]^3$$

Denote by $\tilde{w}_N^\alpha\in[E_N]^3$ and $p_N^\alpha\in E_N$ the numerical approximations to $\tilde{w}$ and p, respectively, at time $\tau_\alpha=\alpha\Delta t$, for $\alpha\in\mathbb{Z}$. Our first scheme is the following:

Algorithm 1

Initialization:

$$\tilde{w}_N^0\in[E_N]^3 \text{ given}$$

Time step: For $0\leqslant\tau_\alpha\leqslant T$, compute

$$\tilde{w}_N^{\alpha+1}\in[E_N]^3, \quad p_N^{\alpha+1/2}\in E_N, \quad \text{and} \quad \rho^{\alpha+1}\in\mathbb{R}^3 \text{ by:}$$

$$\left.\begin{aligned} -\Delta_C p_N^{\alpha+1/2}&=\nabla\cdot\{C[P_N(\tilde{w}_N^{\alpha+1/2}\times\tilde{r}_N^{\alpha+1/2})]\} \\ \langle p_N^{\alpha+1/2}\rangle&=0\end{aligned}\right\} \tag{12}$$

$$\left.\begin{aligned}&\frac{\tilde{w}_N^{\alpha+1}-\tilde{w}_N^{\alpha}}{\Delta t}=C[P_N(\tilde{w}_N^{\alpha+1/2}\times\tilde{r}_N^{\alpha+1/2})+\nabla p_N^{\alpha+1/2}]+\rho^{\alpha+1}\\&\langle\tilde{w}_N^{\alpha+1}\rangle=0\end{aligned}\right\}\tag{13}$$

where

$$\tilde{w}_N^{\alpha+1/2}=\frac{1}{2}(\tilde{w}_N^{\alpha+1}+\tilde{w}_N^{\alpha}),\quad \tilde{r}_N^{\alpha+1/2}=\nabla\times(C^{-1}\tilde{w}_N^{\alpha+1/2})$$

and Δ_C is the linear elliptic operator given by:

$$\Delta_C=\sum_{k,l=1}^{3}C_{kl}\frac{\partial^2}{\partial y_k\,\partial y_l}$$

Remarks

- System (12)–(13) defines $\tilde{w}_N^{\alpha+1}$ implicitly in terms of $\tilde{w}_N^{\alpha}$, with $\rho^{\alpha+1}\in\mathbb{R}^3$ determined by the restriction $\langle\tilde{w}_N^{\alpha+1}\rangle=0$. This system has a solution for any value (positive or negative) of Δt, by virtue of Brouwer's fixed point theorem (see COROLLARY 1). This is used to define our discrete solution up to any preset time $T>0$.
- Equation (13) is close, but it is not exactly, Crank–Nicholson's scheme applied to the time solution of (11).

We shall describe now the properties of Algorithm 1.

LEMMA 1 (i). The sequence of functions $\{\tilde{w}_N^{\alpha}\}$ defined by Algorithm 1 verifies:

$$\nabla\cdot\tilde{w}_N^{\alpha}=\nabla\cdot\tilde{w}_0^N,\ \forall\alpha\in\mathbb{Z}$$

(ii) Assume $\nabla\cdot\tilde{w}_0^N=0$. Then, if (12) holds, problem (13) is equivalent to the following set of equations:

$$\frac{\tilde{r}_N^{\alpha+1}-\tilde{r}_N^{\alpha}}{\Delta t}-\nabla\times[P_N(\tilde{w}_N^{\alpha+1/2}\times\tilde{r}_N^{\alpha+1/2})]=0\tag{14}$$

$$\tilde{w}_N^{\alpha+1}=\nabla\times(C^{-1}\tilde{\psi}_N^{\alpha+1})\tag{15}$$

$$\left.\begin{aligned}&-\Delta_C\tilde{\psi}_N^{\alpha+1}=\tilde{r}_N^{\alpha+1}\\&\langle\tilde{\psi}_N^{\alpha+1}\rangle=0,\psi_N^{\alpha+1}\text{ periodic with period cell }Y\end{aligned}\right\}\tag{16}$$

Proof

Note at first that after discretization, all functions we are dealing with are of C^∞ class, so that we may consider all derivatives of all orders in classical sense.

(i)

This is a direct consequence of (12).

(ii)

Equation (14) follows directly from (13). To prove (15), observe at first that problem (16) has a unique solution $\tilde{\psi}_N^{\alpha+1}\in[E_N]^3$. Observe also that $\nabla\cdot\tilde{\psi}_N^{\alpha+1}$ is periodic with period cell Y, and that $-\Delta_C(\nabla\cdot\tilde{\psi}_N^{\alpha+1})=0$; $\langle\nabla\cdot\tilde{\psi}_N^{\alpha+1}\rangle=0$. Then, $\nabla\cdot\tilde{\psi}_N^{\alpha+1}=0$.

Let us define $\tilde{v}_N=\nabla\times(C^{-1}\tilde{\psi}_N^{\alpha+1})$. Now, the formula:

$$\nabla\times\{C^{-1}[\nabla\times(C^{-1}\tilde{\psi})]\}=-\Delta_C\tilde{\psi}+(C\nabla)(\nabla\cdot\tilde{\psi})\tag{17}$$

yields $\nabla\times(C^{-1}\tilde{v}_N)=-\Delta_C\tilde{\psi}_N^{\alpha+1}=\tilde{r}_N^{\alpha+1}$. Also, formula (17) applied to $\tilde{z}_N^{\alpha}=\tilde{w}_N^{\alpha+1}-\tilde{v}_N$ yields: $\Delta_C\tilde{z}_N^{\alpha}=0$. Also, we have $\langle\tilde{z}_N^{\alpha}\rangle=0$, $\tilde{z}_N^{\alpha}$ periodic with period cell Y. Consequently, $\tilde{z}_N^{\alpha}=0$ and (15) holds.

To prove now that (13) follows from (14), (15) and (16), note at first that (15), (16) and (17) imply:

$$\tilde{r}_N^{\alpha+1}=\nabla\times(C^{-1}\tilde{w}_N^{\alpha+1})$$

Thus, (14) yields:

$$\nabla\times(C^{-1}\tilde{\zeta}_N^{\alpha})=0$$

with

$$\tilde{\zeta}_N^{\alpha}=\frac{\tilde{w}_N^{\alpha+1}-\tilde{w}_N^{\alpha}}{\Delta t}-C[P_N(\tilde{w}_N^{\alpha+1/2}\times\tilde{r}_N^{\alpha+1/2})+\nabla p_N^{\alpha+1/2}]$$

By virtue of (12) and the hypothesis $\nabla\cdot\tilde{w}_0^N$, $\nabla\cdot\tilde{\zeta}_N^{\alpha}=0$. Thus, (15) and (17) yield $\Delta_C\tilde{\zeta}_N^{\alpha}=0$. As $\tilde{\zeta}_N^{\alpha}\in[E_N]^3$, the only solution to this equation is a constant, which is given by the condition $\langle\tilde{w}_N^{\alpha+1}\rangle=0$ deduced from (15). ■

Remark

The condition $\nabla\cdot\tilde{w}_0^N=0$ is obtained, for instance, if $\tilde{w}_0^N$ is the L^2 orthogonal projection of the initial condition $\tilde{w}_0$ onto $[E_N]^3$. Indeed, as the functions ϕ_k are eigenfunctions of all derivatives, the orthogonal projection commutes with the derivatives.

The conservation properties of Algorithm 1 are stated as follows:

THEOEREM 1 Let us denote by q^{α} and h^{α} the mean kinetic energy and the mean helicity of $\tilde{w}_N^{\alpha}$, respectively; i.e. $q^{\alpha}=\frac{1}{2}\langle\tilde{w}_N^{\alpha}\cdot C^{-1}\tilde{w}_N^{\alpha}\rangle$, $h^{\alpha}=\frac{1}{2}\langle\tilde{w}_N^{\alpha}\cdot C^{-1}\tilde{r}_N^{\alpha}\rangle$. Then, $q^{\alpha}=q^0$, $h^{\alpha}=h^0$, $\forall^{\alpha}\in\mathbb{Z}$.

Proof

From (14) we obtain at first:

$$(\tilde{r}_N^{\alpha+1},\varphi_N)=(\tilde{r}_N^{\alpha},\varphi_N)+\Delta t(\nabla\times[P_N(\tilde{w}_N^{\alpha+1/2}\times\tilde{r}_N^{\alpha+1/2})],\varphi_N),\quad\forall\varphi\in[E_N]^3 \tag{18}$$

where $(\cdot,\cdot)$ represents the scalar product on $[L^2(Y,\mathbb{C})]^3$:

$$(\varphi_1,\varphi_2)=\int_Y\varphi_1(y)\cdot\bar{\varphi}_2(y)\,\mathrm{d}y,\quad\forall\varphi_1,\varphi_2\in[L^2(Y,\mathbb{C})]^3 \tag{19}$$

Let us define the following discrete scalar product on $[C_p(Y,\mathbb{C})]^3$:

$$(\varphi_1,\varphi_2)_N=\frac{1}{(2N)^3}\sum_{j_1=0}^{2N-1}\sum_{j_2=0}^{2N-1}\sum_{j_3=0}^{2N-1}\varphi_1(\mathrm{j}h)\cdot\bar{\varphi}_2(\mathrm{j}h) \tag{20}$$

Observe that the quadrature formula

$$\int_0^{2\pi}f(\xi)\,\mathrm{d}\xi=\frac{1}{2N}\sum_{j=0}^{N-1}f\left(\frac{2\pi}{2N}\right)j$$

is exact on S_{2N-1}[15]. Then, the discrete scalar product (20) equals the continuous one (19) in particular if φ_1, φ_2 are functions of $[E_N]^3$ such that one of them at least is real.

Integration by parts yields:

$$\begin{aligned}(\nabla\times[P_N(\tilde{w}_N^{\alpha+1/2}\times\tilde{r}_N^{\alpha+1/2})],\varphi_N)&=(P_N(\tilde{w}_N^{\alpha+1/2}\times\tilde{r}_N^{\alpha+1/2}),\nabla\times\varphi_N)\\&=(P_N(\tilde{w}_N^{\alpha+1/2}\times\tilde{r}_N^{\alpha+1/2}),\nabla\times\varphi_N)_N\\&=(\tilde{w}_N^{\alpha+1/2}\times\tilde{r}_N^{\alpha+1/2},\nabla\times\varphi_N)_N,\quad\forall\varphi_N\in[E_N]^3\end{aligned} \tag{21}$$

Equation (18) reads now:

$$(\tilde{r}_N^{\alpha+1},\varphi_N)=(\tilde{r}_N^{\alpha},\varphi_N)+\Delta t(\tilde{w}_N^{\alpha+1/2}\times\tilde{r}_N^{\alpha+1/2},\nabla\times\varphi_N)_N,\quad\forall\varphi_N\in[E_N]^3 \tag{22}$$

This formulation is used to prove the conservation of q^α and h^α with ease. Indeed, taking successively $\varphi_N = C^{-1}\tilde{\psi}_N^{\alpha+1} \in E_N^3$ and $\varphi_N = C^{-1}\tilde{\psi}_N^{\alpha} \in E_N^3$ yields:

$$(\tilde{w}_N^{\alpha+1}, C^{-1}\tilde{w}_N^{\alpha+1}) = (\tilde{r}_N^\alpha, C^{-1}\tilde{\psi}_N^{\alpha+1}) + \frac{\Delta t}{2}(\tilde{w}_N^{\alpha+1} \times \tilde{w}_N^\alpha, \tilde{r}_N^{\alpha+1/2})_N$$

$$(\tilde{w}_N^\alpha, C^{-1}\tilde{w}_N^\alpha) = (\tilde{r}_N^{\alpha+1}, C^{-1}\tilde{\psi}_N^\alpha) + \frac{\Delta t}{2}(\tilde{w}_N^\alpha \times \tilde{w}_N^{\alpha+1}, \tilde{r}_N^{\alpha+1/2})_N$$

As matrix C is symmetric, integration by parts yields:

$$\begin{aligned}(\tilde{r}_N^\alpha, C^{-1}\tilde{\psi}_N^{\alpha+1}) &= (C^{-1}\tilde{w}_N^\alpha, \nabla \times (C^{-1}\tilde{\psi}_N^{\alpha+1})) = (C^{-1}\tilde{w}_N^\alpha, \tilde{w}_N^{\alpha+1})\\ &= (\tilde{w}_N^\alpha, C^{-1}\tilde{w}_N^{\alpha+1}) = (\tilde{r}_N^{\alpha+1}, C^{-1}\tilde{\psi}_N^\alpha)\end{aligned}$$

Then, $q^{\alpha+1} = q^\alpha$.

Also, to prove the conservation of the mean helicity, one should consecutively take $\varphi_N = C^{-1}\tilde{w}_N^\alpha$ and $\varphi_N = C^{-1}\tilde{w}_N^{\alpha+1}$ in (22). ■

Remark

Although Crank–Nicholson's scheme is frequently used in flow problems to build up numerical algorithms with good conservation properties, this is not the case here. One may be convinced of this fact by reproducing the proof above for Crank–Nicholson's scheme. Also, leap-frog scheme is well known to provide conservative discretizations in flow problems[21]. However, it can be proved with a little more work that leap-frog scheme again fails to reproduce the conservation properties of Theorem 1. Algorithm 1 seems to be especially well suited to conserve kinetic energy and helicity for Euler equations.

The conservation of kinetic energy allows to ensure the existence of solutions of Algorithm 1, as follows:

COROLLARY 1 System (12)–(13) has always at least one solution.

Proof

Given $\tilde{w}_N^\alpha \in [E_N]^3$, consider the transformation $\mathscr{T}: [E_N]^3 \to [E_N]^3$ given by:

$$\mathscr{T}(\tilde{w}) = \tilde{w}_N^\alpha + \Delta t C[P_N(\tilde{z} \times \tilde{r}) + \nabla p] + \tilde{\rho}$$

where $\tilde{z} = \frac{1}{2}(\tilde{w} + \tilde{w}_N^\alpha)$, $\tilde{r} = \nabla \times (C^{-1}\tilde{z})$, and p is the only solution of:

$$\left.\begin{aligned} -\Delta_C p &= \nabla \cdot \{C[P_N(\tilde{z} \times \tilde{r})]\}\\ p \in E_N, &\quad \langle p \rangle = 0\end{aligned}\right\}$$

and

$$\tilde{\rho} = -\langle \tilde{w}_N^\alpha + \Delta t C[P_N(\tilde{z} \times \tilde{r}) + \nabla p]\rangle \in \mathbb{R}^3$$

From Theorem 1, we obtain

$$|||\mathscr{T}(\tilde{w})||| = |||\tilde{w}_N^\alpha|||$$

where

$$|||\mathscr{T}||| = \frac{1}{2}\langle \tilde{w} \cdot C^{-1}\tilde{w}\rangle$$

which is a norm on $[L^2(Y)]^3$.

As $\mathscr{T}$ is continuous, the Corollary follows from Brouwer's fixed point theorem. ■

Remark

Observe that for each $\alpha \in \mathbb{Z}^3$ there exists Δt_α depending on $\tilde{w}_N^\alpha$ such that $\mathscr{T}$ is contractive if $\Delta t < \Delta t_\alpha$. Thus, for such Δt the solution of (12)–(13) is unique. However, unicity is not ensured up to any arbitrary time T, as Δt_α can decrease arbitrarily fast.

It is possible to introduce a slight modification of Algorithm 1 in such a way that the mean of $\tilde{w}_N^\alpha$ is also conserved. To do so, one should replace the discrete L^2 orthogonal projection P_N by the continuous one Q_N. This yields the following algorithm.

Algorithm 2

Initialization

$$\tilde{w}_N^0 \in [E_N]^3 \text{ given}$$

Time step: For $0 \leqslant \tau_\alpha \leqslant T$, compute

$$\tilde{w}_N^{\alpha+1} \in [E_N]^3 \quad \text{and} \quad p_N^{\alpha+1/2} \in E_N, \quad \text{by}$$

$$\left.\begin{aligned} -\Delta_C p_N^{\alpha+1/2} &= \nabla\cdot\{C[Q_N(\tilde{w}_N^{\alpha+1/2} \times \tilde{r}_N^{\alpha+1/2})]\} \\ \langle p_N^{\alpha+1/2}\rangle &= 0 \end{aligned}\right\} \tag{23}$$

$$\left.\frac{\tilde{w}_N^{\alpha+1} - \tilde{w}_N^\alpha}{\Delta t} = C[Q_N(\tilde{w}_N^{\alpha+1/2} \times \tilde{r}_N^{\alpha+1/2}) + \nabla p_N^{\alpha+1/2}]\right\} \tag{24}$$

The conservation properties of Alg 2 are stated as follows.

THEOREM 2 Let us denote by q^α and h^α the mean kinetic energy and the mean helicity, respectively, of the sequence $\{\tilde{w}_N^\alpha\}_{\alpha\in\mathbb{Z}}$ furnished by Algorithm 2. Then,

$$\langle \tilde{w}_N^\alpha \rangle = 0, \quad q^\alpha = q^0, \quad h^\alpha = h^0, \quad \forall^\alpha \in \mathbb{Z}$$

Proof

Let us observe that from (10) we may write:

$$\begin{aligned} \tilde{w}_N^{\alpha+1/2} \times \tilde{r}_N^{\alpha+1/2} &= \frac{1}{2}\nabla|G^{-\mathrm{T}}\tilde{w}_N^{\alpha+1/2}|^2 - C^{-1}(\tilde{w}_N^{\alpha+1/2}\cdot\nabla\tilde{w}_N^{\alpha+1/2}) \\ &= \frac{1}{2}\nabla|G^{-\mathrm{T}}\tilde{w}_N^{\alpha+1/2}|^2 - C^{-1}[\nabla\cdot(\tilde{w}_N^{\alpha+1/2} \otimes \tilde{r}_N^{\alpha+1/2})] \end{aligned}$$

Then,

$$\left(\frac{2\pi}{s}\right)^3 \langle Q_N(\tilde{w}_N^{\alpha+1/2} \times \tilde{r}_N^{\alpha+1/2})\rangle = (Q_N(\tilde{w}_N^{\alpha+1/2} \times \tilde{r}_N^{\alpha+1/2}), 1) = (\tilde{w}_N^{\alpha+1/2} \times \tilde{r}_N^{\alpha+1/2}, 1) = 0$$

Consequently, (24) yields:

$$\langle \tilde{w}_N^{\alpha+1} \rangle = \langle \tilde{w}_N^\alpha \rangle$$

The conservation of q and h are proved exactly as in Theorem 1. ■

Remarks

- Corollary 1 applies also to Algorithm 2: System (23)–(24) has always a solution.
- Note that if the $\tilde{w}_N^\alpha$ are uniformly bounded in H^1 norm, then the mapping $\mathscr{T}$ of Corollary 1 is uniformly contractive, and the unicity of $\tilde{w}_N^\alpha$ is ensured. This is the case when Algorithm 2 is applied to two-dimensional flows. We state this result as follows

COROLLARY 2 Assume $\tilde{w}_N^0$ and the matrix C are two-dimensional, in the sense that they have the following structure:

$$\tilde{w}_N^0(y)=(w_{N1}^0(y_1, y_2), w_{N2}^0(y_1, y_2), 0); \quad C=\begin{bmatrix} c_{11} & c_{12} & 0 \\ c_{12} & c_{22} & 0 \\ 0 & 0 & c_{33} \end{bmatrix}$$

Then, the following holds:

(i) The discrete enstropy is conserved:

$$\|\tilde{r}_N^{\alpha+1}\|_{0,2}^2=\|\tilde{r}_N^0\|_{0,2}^2, \quad \forall\alpha\in\mathbb{Z}$$

(ii) There exists a Δt_0 depending only on $\tilde{w}_n^0$ such that if $0<\Delta t<\Delta t_0$, the solution of (23)–(24) is unique.

Proof

(i) To prove that the L^2 norm of $\tilde{r}_N^\alpha$ is conserved constant, observe at first that formula (10) yields:

$$\tilde{w}_N^{\alpha+1}=\tilde{w}_N^\alpha+\Delta t[-Q_N(\tilde{w}_N^{\alpha+1/2}\cdot\nabla\tilde{w}_N^{\alpha+1/2})+C\nabla\pi_N^{\alpha+1/2}]$$

where $\pi_N^{\alpha+1/2}\in E_N$ is the solution of:

$$\Delta_C\pi_N^{\alpha+1/2}=\nabla\cdot[Q_N(\tilde{w}_N^{\alpha+1/2}\cdot\nabla\tilde{w}_N^{\alpha+1/2})]; \quad \langle\pi_N^{\alpha+1/2}\rangle=0$$

Then, if $\tilde{w}_N^\alpha$ does not depend on y_3, the same holds for $\pi_N^{\alpha+1/2}$ and $\tilde{w}_N^{\alpha+1}$. Furthermore, due to the structure above of matrix C, a solution of (12) corresponds necessarily to $\tilde{w}_{N3}=0$. Then, at any time step the velocity field $\tilde{w}_N^\alpha$ has the same two-dimensional structure as $\tilde{w}_N^0$.

Consider now that

$$\nabla\times[C^{-1}(\tilde{w}\cdot\nabla\tilde{w})]=\tilde{w}\cdot\nabla\tilde{r}-\tilde{r}\cdot\nabla\tilde{w}$$

and thus (14) may be written as:

$$\frac{\tilde{r}_N^{\alpha+1}-\tilde{r}_N^\alpha}{\Delta t}=Q_N(\tilde{r}_N^{\alpha+1/2}\cdot\nabla\tilde{w}_N^{\alpha+1/2}-\tilde{w}_N^{\alpha+1/2}\cdot\nabla\tilde{r}_N^{\alpha+1/2})$$

Due to the structures of $\tilde{w}_N^\alpha$ and C, we have $\tilde{r}_N^{\alpha+1/2}\cdot\nabla\tilde{w}_N^{\alpha+1/2}=0$, and then:

$$(\tilde{r}_N^{\alpha+1}-\tilde{r}_N^\alpha, \tilde{r}_N^{\alpha+1}+\tilde{r}_N^\alpha)=-2\Delta t(\tilde{w}_N^{\alpha+1/2}\cdot\nabla\tilde{r}_N^{\alpha+1/2}, \tilde{r}_N^{\alpha+1/2})=\Delta t(\nabla\cdot\tilde{w}_N^{\alpha+1/2}, \|\tilde{r}_N^{\alpha+1/2}\|^2)=0$$

Finally, this yields:

$$\|\tilde{r}_N^{\alpha+1}\|_{0,2}^2=\|\tilde{r}_N^\alpha\|_{0,2}^2$$

(ii) In the same way as in (i) one proves that the mapping $\mathscr{T}$ of Corollary 1 conserves kinetic energy and enstropy. Thus, given $\tilde{w}_1, \tilde{w}_2\in[E_N]^3$, we have:

$$\|\mathscr{T}(\tilde{w}_1)-\mathscr{T}(\tilde{w}_2)\|_{0,2}\leqslant\frac{\Delta t}{4}(\|\tilde{w}_N^\alpha\|_{0,2}+\|\tilde{r}_N^\alpha\|_{0,2})(\|\tilde{w}_1-\tilde{w}_2\|_{0,2}+\|\tilde{r}_1-\tilde{r}_2\|_{0,2})\leqslant K_N\|\tilde{w}_1-\tilde{w}_2\|_{0,2}$$

where K_N is a constant depending on $\tilde{w}_N^0$ and N.

Thus, $\mathscr{T}$ is contractive for Δt small enough, uniformly in α. ■

Computational aspects

Let us remark that the coefficients $\hat{v}_k$ that define the interpolation operator P_N may be computed in $O(N^3 \log N)$ operations, by means of the Fast Fourier Transform (FFT). It is also possible to compute the values of $P_N v$ at the nodes (jh) starting from the $\hat{v}_k$ in $O(N^3 \log N)$ by means of the inverse FFT[1]. A direct calculation would need $O(N^6)$ operations in both cases.

Let us remark also that the orthogonal projection of $\tilde{w}_N^{\alpha+1/2}\times\tilde{r}_N^{\alpha+1/2}$ onto $[E_N]^3$ can be computed exactly by means of the FFT with $2N$ points. Indeed,

$$Q_N(\tilde{w}_N^{\alpha+1/2}\times\tilde{r}_N^{\alpha+1/2})=\sum_{k_1=-N}^{N-1}\sum_{k_2=-N}^{N-1}\sum_{k_3=-N}^{N-1}\hat{c}_{\mathbf{k}}\phi_{\mathbf{k}}$$

where the coefficients $\hat{c}_{\mathbf{k}}$ are given by:

$$\hat{c}_{\mathbf{k}}=(\tilde{w}_N^{\alpha+1/2}\times\tilde{r}_N^{\alpha+1/2},\ \phi_{\mathbf{k}})$$

Moreover, as we have stated, if φ_1, φ_2 are functions of $[E_{2N}]^3$ and one of them at least is real, then[15]:

$$(\varphi_1,\ \varphi_2)_{2N}=(\varphi_1,\ \varphi_2)$$

Consequently,

$$\hat{c}_{\mathbf{k}}=\frac{1}{(4N)^3}\sum_{j_1=0}^{4N-1}\sum_{j_2=0}^{4N-1}\sum_{j_3=0}^{4N-1}(\tilde{w}_N^{\alpha+1/2}\times\tilde{r}_N^{\alpha+1/2})(\mathbf{j}h)\bar{\phi}_{\mathbf{k}}(\mathbf{j}h)$$

$$\forall\mathbf{k}=(k_1,\ k_2,\ k_3)\quad\text{such that}\quad k_1,\ k_2,\ k_3=-2N,\ldots,2N-1$$

Then, the coefficients $\hat{c}_{\mathbf{k}}$ may be computed with the FFT with $2N$ points.

CONVERGENCE ANALYSIS

It is well known that there are no results of existence of solutions of the Cauchy problem for Euler equations (3) for arbitrarily large time intervals. However, implicit function theory allows to prove the existence of smooth solutions for short time intervals[2,14]. Our next results state the convergence of our algorithms during the time interval of existence of smooth solutions, under some restrictions on the behaviour of the H^1 norm of the discrete solutions.

Let us at first recall the error estimates concerning the interpolation error associated to Q_N[15].

LEMMA 2 Assume $z\in H_P^l(Y)$ with $l\geqslant 0$. Then, there exists a constant K depending only on z such that:

(i)

$$\|z-Q_Nz\|_{m,2}\leqslant\frac{K}{N^{l-m}}\|z\|_{l,2}$$

(ii)

$$\|Q_Nz\|_{m,2}\leqslant K\|z\|_{l,2}$$

for all $m=0,\ 1,\ldots,l$.

■

Our main convergence result is stated as follows.

THEOREM 3 Let us assume that problem (11) admits a unique solution during a time interval $[0,\ T]$, verifying:

$$\tilde{w}\in C^3(0,\ T;[W_P^{k,\infty}(Y)]^3)\quad\text{with } k\geqslant 2$$

Assume also that the initial condition $\tilde{w}_N^0$ verifies:

$$\|\tilde{w}^0-\tilde{w}_N^0\|_{0,2}\leqslant\frac{C}{N^{k-1}}\tag{25}$$

for some constant C_0.

Finally, assume that the sequence $\{\tilde{w}_N^\alpha\}_{\alpha\in\mathbf{Z}}$ provided by Algorithm 2 is bounded in $[H^1(Y)]^3$.

Then, the sequence $\{\tilde{w}_N^\alpha\}_{\alpha\in\mathbf{Z}}$ converges to the solution $\tilde{w}$ of problem (11) in $[L^2(Y)]^3$, uniformly in time. More precisely, there exists a constant c_T depending on $\tilde{w}_0$ and the time length T, such that:

$$\max_{0\leq t_\alpha\leq T} \|\tilde{w}(.,t_\alpha)-\tilde{w}_N^\alpha\|_{0,2}\leq c_T\left(\frac{1}{N^{k-1}}+\Delta t^2\right) \tag{26}$$

In particular, the convergence in L^2 norm is second order accurate if $k\geqslant 3$.

Proof

Let us recall that by virtue of Corollary 1, Algorithm 2 provides at least a sequence $\{\tilde{w}_N^\alpha\}_{\alpha\in\mathbf{Z}}$ for each fixed $\Delta t>0$.

Let us denote $\tilde{w}(.,t_\alpha)$ by $\tilde{w}^\alpha$ and $\tilde{r}(.,t_\alpha)$ by $\tilde{r}^\alpha$, and define the discretization error on $\tilde{w}^\alpha$ by:

$$\tilde{e}^\alpha=\tilde{w}^\alpha-\tilde{w}_N^\alpha$$

Consistency estimate

Let us define the consistency error by:

$$\tilde{\eta}^\alpha=\tilde{w}^{\alpha+1}-\tilde{w}^\alpha-\Delta t C[Q_N(\tilde{w}^{\alpha+1/2}\times\tilde{r}^{\alpha+1/2})+\nabla\tilde{p}^{\alpha+1/2}] \tag{27}$$

where $\tilde{p}^{\alpha+1/2}\in H_P^1(Y)$ is the solution of:

$$\left.\begin{array}{l}-\Delta_C\tilde{p}^{\alpha+1/2}=\nabla\cdot\{C[Q_N(\tilde{w}^{\alpha+1/2}\times\tilde{r}^{\alpha+1/2})]\}\\ \langle\tilde{p}^{\alpha+1/2}\rangle=0\end{array}\right\}$$

Taylor series yields:

$$\tilde{w}^{\alpha+1/2}\times\tilde{r}^{\alpha+1/2}=\tilde{w}_{,t}(\tau_\alpha)+\frac{\Delta t}{2}\tilde{w}_{,tt}(\tau_\alpha)+C\nabla q_1^\alpha+\Delta t^2R_1^\alpha$$

where $\tau_\alpha=\alpha\Delta t$, q_1^α is a smooth pressure and R_1^α is an algebraic expression depending on time derivatives of orders 0, 1 and 2 of $\tilde{w}$ and $\tilde{r}$. Thus,

$$\tilde{\eta}^\alpha=\Delta t^3R_2^\alpha+C\nabla q_2^\alpha+\Delta t(I-Q_N)(\tilde{w}^{\alpha+1/2}\times\tilde{r}^{\alpha+1/2})$$

where q_2^α is a smooth pressure, and R_2^α is an algebraic expression depending on time derivatives of orders 0, 1 and 2 of $\tilde{w}$ and $\tilde{r}$, and of order 3 of w.

Given $\varphi_N\in[E_N]^3$ with free divergence, we have:

$$|(\tilde{\eta}^\alpha,C^{-1}\varphi_N)|=\leqslant c_1\Delta t\left(\Delta t^2+\frac{1}{N^{k-1}}\right)\|\varphi_N\|_{0,2}$$

for some constant c_1 depending only on $\tilde{w}$ and T.

Stability estimate

Given $\varphi_N\in[E_N]^3$, with free divergence, from (24) we obtain:

$$(C^{-1}(\tilde{w}_N^{\alpha+1}-\tilde{w}_N^\alpha),\varphi_N)=\Delta t(\tilde{w}_N^{\alpha+1/2}\times\tilde{r}_N^{\alpha+1/2},\varphi_N)$$

Also, (27) yields in the same way:

$$(C^{-1}(\tilde{w}^{\alpha+1}-\tilde{w}^\alpha),\varphi_N)=\Delta t(\tilde{w}^{\alpha+1/2}\times\tilde{r}^{\alpha+1/2},\varphi_N)+(\tilde{\eta}^\alpha,C^{-1}\varphi_N)$$

From these last two equations we get:

$$\begin{aligned}(C^{-1}(\tilde{e}^{\alpha+1}-\tilde{e}^\alpha),\varphi_N)=\Delta t[(\tilde{w}^{\alpha+1/2}\times(\tilde{r}^{\alpha+1/2}-\tilde{r}_N^{\alpha+1/2}),\varphi_N)+(\tilde{e}^{\alpha+1/2}\times\tilde{r}_N^{\alpha+1/2},\varphi_N)]+\\ (\tilde{\eta}^\alpha,C^{-1}\varphi_N),\ \forall\varphi_N\in[E_N]^3\text{ such that }\nabla\cdot\varphi_N=0\end{aligned} \tag{28}$$

Let us now define the errors:

$$\tilde{\rho}^\alpha = Q_N \tilde{w}^\alpha - \tilde{w}_N^\alpha, \quad \tilde{\varepsilon} = \tilde{e}^\alpha - \tilde{\rho}^\alpha = \tilde{w}^\alpha - Q_N \tilde{w}^\alpha$$

and observe that $\nabla \cdot \tilde{\rho}^\alpha = \nabla \cdot \tilde{\varepsilon}^\alpha = 0$, because the operator Q_N conmutes with the derivatives. Observe also that:

$$(C^{-1}(\tilde{e}^{\alpha+1} - \tilde{e}^\alpha), \tilde{\rho}^{\alpha+1/2}) = \frac{1}{2}[(C^{-1}\tilde{\rho}^{\alpha+1}, \tilde{\rho}^{\alpha+1}) - (C^{-1}\tilde{\rho}^\alpha, \tilde{\rho}^\alpha)] + (C^{-1}(\tilde{\varepsilon}^{\alpha+1} - \tilde{\varepsilon}^\alpha), \tilde{\rho}^{\alpha+1/2}) \quad (29)$$

Moreover,

$$\tilde{\varepsilon}^{\alpha+1} - \tilde{\varepsilon}^\alpha = (I - Q_N)(\tilde{w}_N^{\alpha+1} - \tilde{w}_N^\alpha) = \Delta t (I - Q_N)[\tilde{w}_{,t}(\tau_\alpha + \theta_\alpha)], \quad \text{for some } \theta_\alpha \in]0, 1[$$

Then, there exists a constant c_2 depending on T such that:

$$\| \tilde{\varepsilon}^{\alpha+1} - \tilde{\varepsilon}^\alpha \|_{0,2} \leqslant c_2 \frac{\Delta t}{N^k}, \quad 0 \leqslant \tau_\alpha \leqslant T \quad (30)$$

This suggests to take $\varphi_N = \tilde{\rho}^{\alpha+1/2}$ in (28). Let us obtain estimates for the r.h.s. of (28) in this case.

$$\begin{aligned}(\tilde{w}^{\alpha+1/2} \times (\tilde{r}^{\alpha+1/2} - \tilde{r}_N^{\alpha+1/2}), \tilde{\rho}^{\alpha+1/2}) = {} & ([\nabla \times (C^{-1}\tilde{\rho}^{\alpha+1/2})] \times \tilde{\rho}^{\alpha+1/2}, \tilde{w}^{\alpha+1/2}) \\ & + (\tilde{w}^{\alpha+1/2} \times [\nabla \times (C^{-1}\tilde{\varepsilon}^{\alpha+1/2})], \tilde{\rho}^{\alpha+1/2}) \end{aligned} \quad (31)$$

Formula (10) yields:

$$[\nabla \times (C^{-1}\tilde{\rho}^{\alpha+1/2})] \times \tilde{\rho}^{\alpha+1/2} = C^{-1}\nabla \cdot (\tilde{\rho}^{\alpha+1/2} \otimes \tilde{\rho}^{\alpha+1/2}) - \frac{1}{2}\nabla |G^{-T}\tilde{\rho}^{\alpha+1/2}|^2$$

Thus

$$\begin{aligned}([\nabla \times (C^{-1}\tilde{\rho}^{\alpha+1/2})] \times \tilde{\rho}^{\alpha+1/2}, \tilde{w}^{\alpha+1/2}) = {} & \frac{1}{2}(|G^{-T}\tilde{\rho}^{\alpha+1/2}|^2, \nabla \cdot \tilde{w}^{\alpha+1/2}) - \\ & (\tilde{\rho}^{\alpha+1/2} \otimes \tilde{\rho}^{\alpha+1/2}, \nabla(C^{-1}\tilde{w}^{\alpha+1/2}))\end{aligned}$$

This, Lemma 3 and (31) yield the following estimate:

$$|(\tilde{w}^{\alpha+1/2} \times (\tilde{r}^{\alpha+1/2} - \tilde{r}_N^{\alpha+1/2}), \tilde{\rho}^{\alpha+1/2}))| \leqslant c_3 \left(\|\tilde{\rho}^{\alpha+1/2}\|_{0,2} + \frac{1}{N^{k-1}} \right) \|\tilde{\rho}^{\alpha+1/2}\|_{0,2} \quad (32)$$

where c_3 is a constant depending only on $\tilde{w}$ and T. Also, we may bound the second term in the r.h.s. of (28) as follows:

$$|(\tilde{w}^{\alpha+1/2} \times (\tilde{r}^{\alpha+1/2} - \tilde{r}_N^{\alpha+1/2}), \tilde{\rho}^{\alpha+1/2})| = |(\tilde{\varepsilon}^{\alpha+1/2} \times \tilde{r}_N^{\alpha+1/2}, \tilde{\rho}^{\alpha+1/2})| \leqslant c_4 \frac{M}{N^k} \|\tilde{\rho}^{\alpha+1/2}\|_{0,2} \quad (33)$$

where M is a uniform bound of the L^2 norm of the $\tilde{r}_N^{\alpha+1/2}$.

Now, (29), (30), (32) and (33) yield:

$$\|\tilde{\rho}^{\alpha+1}\|_{0,2}^2 - \|\tilde{\rho}^\alpha\|_{0,2}^2 \leqslant c_5 \Delta t \left(\|\tilde{\rho}^\alpha\|_{0,2} + \|\tilde{\rho}^{\alpha+1}\|_{0,2} + \frac{1}{N^{k-1}} + \frac{M}{N^k} + \Delta t^2 \right) [\|\tilde{\rho}^{\alpha+1}\|_{0,2} + \|\tilde{\rho}^\alpha\|_{0,2}]$$

and consequently

$$(1 - c_5 \Delta t)\|\tilde{\rho}^{\alpha+1}\|_{0,2} \leqslant (1 + c_5 \Delta t)\|\tilde{\rho}^\alpha\|_{0,2} + c_5 \Delta t \left(\frac{1}{N^{k-1}} + \frac{M}{N^k} + \Delta t^2 \right)$$

Thus, if Δt is small enough,

$$\|\tilde{\rho}^{\alpha+1}\|_{0,2} \leqslant (1 + c_6 \Delta t)\|\tilde{\rho}^\alpha\|_{0,2} + c_6 \Delta t \left(\frac{1}{N^{k-1}} + \frac{M}{N^k} + \Delta t^2 \right)$$

Finally, Gronwall's Lemma yields

$$\|\tilde{\rho}^{\alpha+1}\|_{0,2} \leqslant c_7\left(\|\tilde{\rho}^0\|_{0,2}+\frac{1}{N^{k-1}}+\frac{M}{N^k}+\Delta t^2\right)$$

This proves our Theorem. ■

Remarks

- Algorithm 2 is still convergent if $M=o(N^k)$; i.e. if the H^1 norm of $\tilde{w}_N^\alpha$ explodes slower than N^k.
- The main difficulty in proving the global existence of solutions for 3D Euler equations is to obtain estimates for the H^1 norm of the solution. Thus, it is not surprising our restriction on the behaviour of the H^1 norm of $\tilde{w}_N^\alpha$ to prove the convergence.
- In our numerical experiments, for short times the L^2 norm of $\tilde{r}_N^\alpha$ is not very sensitive to the actual value of N.
- Algorithm 2 always converges if the flow is two-dimensional, in the sense of Corollary 2. Indeed, in this case the L^2 norm of $\tilde{r}_N^\alpha$ is constant.

NUMERICAL EXPERIMENTS

In this section we shall describe some issues concerning the practical implementation of Algorithm 1, so as its practical performance in some test cases.

Initial conditions

As initial condition $\tilde{w}_0$ for system (11) we have taken Beltrami fields:

$$\tilde{w}_0=\begin{bmatrix} a\sin(\lambda y_3)+b\cos(\lambda y_2)\\ c\sin(\lambda y_1)+a\cos(\lambda y_3)\\ b\sin(\lambda y_2)+c\cos(\lambda y_1)\end{bmatrix},\quad \text{for } a, b, c, \lambda\in\mathbb{R} \tag{34}$$

where the parameters a, b, c, λ are chosen in such a way that the value of both the initial mean kinetic energy q^0 and helicity h^0 is one.

Beltrami fields are steady solutions of Euler equations (11) when C is the identity matrix I. Indeed, one has:

$$\nabla\times\tilde{w}_0=\lambda\tilde{w}_0 \quad\text{and then}\quad \tilde{w}_0\times(\nabla\times\tilde{w}_0)=0$$

Thus, in the framework introduced in the statement of the problem, the solution of problem (11) for $C\neq I$ can be viewed as a perturbation produced by the mean flow to an initially steady solution.

Observe that given a solution $\tilde{w}$ of problem (11) with mean kinetic energy q_0 and mean helicity h_0, one always may re-scale $\tilde{w}$ and its period cell to obtain another solution $\tilde{w}'$ with prescribed mean kinetic energy q' and helicity h', as follows:

$$\tilde{w}'(y)=\sqrt{\frac{q'}{q_0}}\,\tilde{w}\left(\frac{h'}{h_0}\frac{q_0}{q'}y\right)$$

Thus, taking $q_0=h_0=1$ is not a real restriction.

We have also made the following choice for matrix C

$$C=\begin{bmatrix} 1+\alpha^2 & \alpha & 0\\ \alpha & 1 & 0\\ 0 & 0 & 1\end{bmatrix},\quad \text{for } \alpha\in\mathbb{R} \tag{35}$$

This matrix corresponds to two-dimensional mean flow u in (4). If the mean flow is three-dimensional, matrix C depends on 5 parameters, so it is a very long lasting task to perform a numerical computation of the closure terms in the general case. Our choice seems to be reasonable to test our code.

Note that for these $\tilde{w}_0$ and C, due to the symmetries of the problem, tensors $\tilde{\mathbf{R}}$ and $\tilde{\mathbf{S}}$ must have the following structure[6,17]:

$$\tilde{\mathbf{R}}=\begin{bmatrix} r_{11} & r_{12} & 0 \\ r_{12} & r_{22} & 0 \\ 0 & 0 & r_{33} \end{bmatrix}, \quad \tilde{\mathbf{S}}=\begin{bmatrix} s_{11} & s_{12} & 0 \\ s_{12} & s_{22} & 0 \\ 0 & 0 & s_{33} \end{bmatrix} \tag{36}$$

Practical issues

We have built up a FORTRAN 77 code that implements Algorithm 1. As we have stated already, we have used the direct and inverse FFT to compute the interpolation operator P_N and its inverse. Also, we have solved the non-linear problem (12)–(13) that defines $\tilde{w}^{\alpha+1}$ by means of a black-box GMRES code[20]. This code provides a fast and highly vectorized solution of this problem. As our turbulence model is fully based upon the theoretical symmetries of Euler equations, it seems appropriate for our purposes to look for numerical solutions that reproduce these symmetries and, in particular, the structures of tensors $\tilde{\mathbf{R}}$ and $\tilde{\mathbf{S}}$. In practice, our numerical solution reproduce approximately these symmetries only if problem (12)–(13) is solved with very high precision (up to an error of order 10^{-9}).

Let us remark also that the closure terms defined in (6) have been computed by means of Simpson's rule. For instance, we have approximated the mean

$$\frac{1}{\mathrm{vol}(Y)2\alpha\,\Delta t}\int_{-\tau_\alpha}^{\tau_\alpha}\int_{Y^3}\tilde{w}\otimes\tilde{w}\,\mathrm{d}y\,\mathrm{d}\tau$$

by

$$\frac{1}{6\alpha}\left[4(\tilde{R}_N^{-\alpha+1}+\tilde{R}_N^{-\alpha+3}+\cdots+\tilde{R}_N^{\alpha-3}+\tilde{R}_N^{\alpha-1})+2(\tilde{R}_N^{-\alpha+2}+\tilde{R}_N^{-\alpha+4}+\cdots+\tilde{R}_N^{\alpha-4}+\tilde{R}_N^{\alpha-2})\right.$$
$$\left.+\tilde{R}_N^{-\alpha}+\tilde{R}_N^{\alpha}\right]$$

where

$$\tilde{R}_N^{\alpha}=\frac{1}{\mathrm{vol}(Y)}\int_Y \tilde{w}_N^{\alpha}\otimes\tilde{w}_N^{\alpha}\,\mathrm{d}y$$

Let us remark that if $\tilde{w}\in W^{1,\infty}(\mathbb{R},[L^2(Y)]^3)$, this formula converges to $\langle\langle\tilde{w}\otimes\tilde{w}\rangle\rangle$ as Δt decreases to zero[3].

Testing of code

To test the computation of the non-linear term $\tilde{w}\times\tilde{r}$ we have taken initial conditions of the form:

$$\tilde{w}_0(y)=w_c\cos(\mathbf{k}\cdot y)+w_s\sin(\mathbf{k}\cdot y) \tag{37}$$

with

$$w_c=\alpha_1\mathbf{u}+\alpha_2\mathbf{k}\times(C^{-1}\mathbf{u}),\quad w_s=\frac{1}{\lambda}[\alpha_2\lambda^2\mathbf{u}+\alpha_1\mathbf{k}\times(C^{-1}\mathbf{u})],\quad \lambda=\sqrt{\mathbf{k}^tC\mathbf{k}}$$

where $k \in \mathbb{Z}^3$ is an arbitrary vector, $\mathbf{u} \in \mathbb{R}^3$ is any vector orthogonal to $\mathbf{k}$, and α_1, α_2 are arbitrary constants. Note that $\tilde{w}_0 \in [E_N]^3$ if $|k_1|, |k_2|, |k_3| \leqslant N-1$.

Such velocity fields are steady solutions of generalized Euler equations (11), verifying:

$$\tilde{r}_0 = \nabla \times (C^{-1}\tilde{w}_0) = \lambda \tilde{w}_0 \Rightarrow \tilde{w}_0 \times \tilde{r}_0 = 0$$

Thus, Algorithm 1 gives $\tilde{w}_N^{\alpha} = \tilde{w}_N^0$, $\forall \alpha \in \mathbb{Z}$. This has just been our result for many choices of the parameters $\mathbf{k}$, $\mathbf{u}$, α_1, α_2, and of the matrix C.

Table 1 Convergence order estimates for statistics computed with Algorithm 1

	$N=8$, $\Delta t=0.6$	$N=16$, $\Delta t=0.3$	$N=32$, $\Delta t=0.15$	$\tilde{p}$
r_{11}	0.667498	0.667544	0.667550	2.93
r_{12}	$-0.140868e-1$	$-0.144872e-1$	$-0.145921e-1$	1.93
r_{22}	0.665116	0.665123	0.665124	2.78
r_{33}	0 663553	0.663578	0.663586	1.63
s_{11}	0.668096	0.668074	0.668070	2.46
s_{12}	$-0.332434e-1$	$-0.332405e-1$	$-0.332398e-1$	2.05
s_{22}	0.664439	0.664434	0.664433	2.30
s_{33}	0.667469	0.667494	0.667501	1.84
ψ_q	2.011851	2.012471	2.012617	2.08
ψ_h	2.007124	2.006848	2.006776	1.93

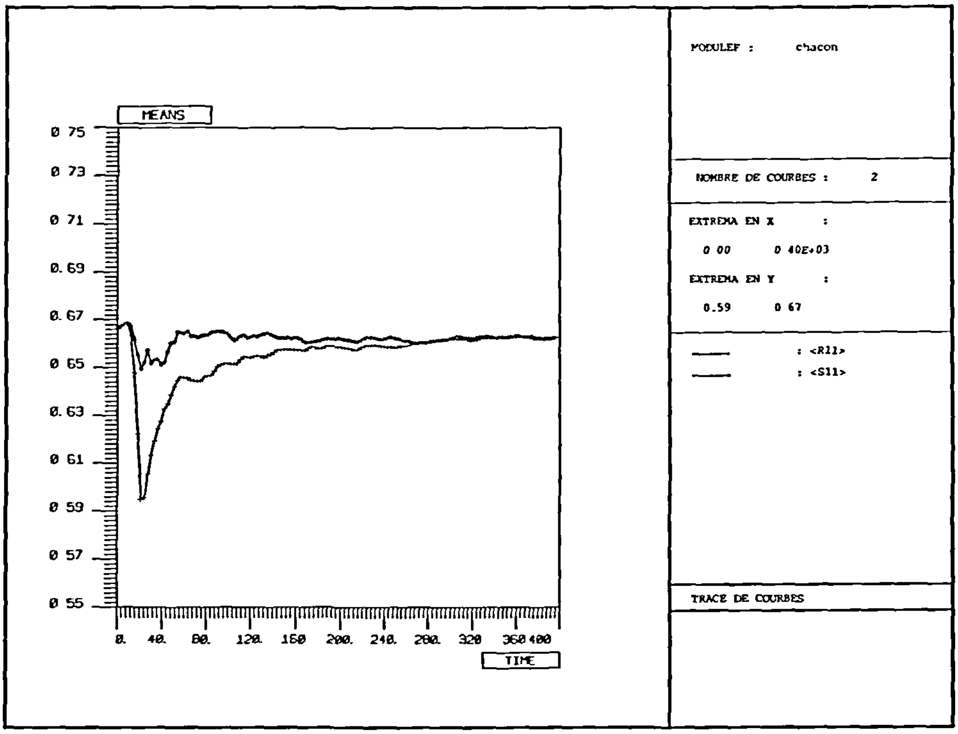

Figure 1 Time evolution of statistics r_{11} and s_{11}, corresponding to the matrix C given by (35) for $\alpha = -0.1$. Both statistics are computed by (37), in the time interval [0, 400]. A quasi-steady state is attempted by time $\tau \simeq 300$

Also, to test the convergence of our algorithm we have compared the asymptotic behaviour of the computed closure terms as Δt and N decrease to zero. We have taken Δt proportional to N, so that there is just one discretization parameter that we call, for instance, h. Let us denote by $e(h)$ the error corresponding to the computation of a given statistic, let us say s, of the solution $\tilde{w}$ by means of Algorithm 1. If we assume that the error e admits an asymptotic expansion of the form:

$$e(h)=\mu h^p+O(h^{p+1})$$

then an estimate of the convergence order of Algorithm 1 is given by:

$$p\simeq\tilde{p}=\log_2\left|\frac{s(2h)-s(h)}{s(h)-s(h/2)}\right|$$

For instance, *Table 1* shows our computed statistics, so as the corresponding estimate convergence order given by the formula above, when $\alpha=-0.1$ in (35), for $\tau=2.4$. The value not represented are of order 10^{-9} or smaller.

This estimation gives orders of convergence that in most cases are quite close to $p=2$, with a certain range of variation. This is probably due to the small differences between the computed statistic in the three cases considered.

Let us remark that the order of accuracy of the interpolation operator P_N is the same as that of Q_N when the interpolated function is smooth enough, in the sense that Lemma 2 is also true if we replace P_N by Q_N, for $l\geqslant 1$. Thus, the order of convergence that could be expected from Theorem 2 is just $p=2$.

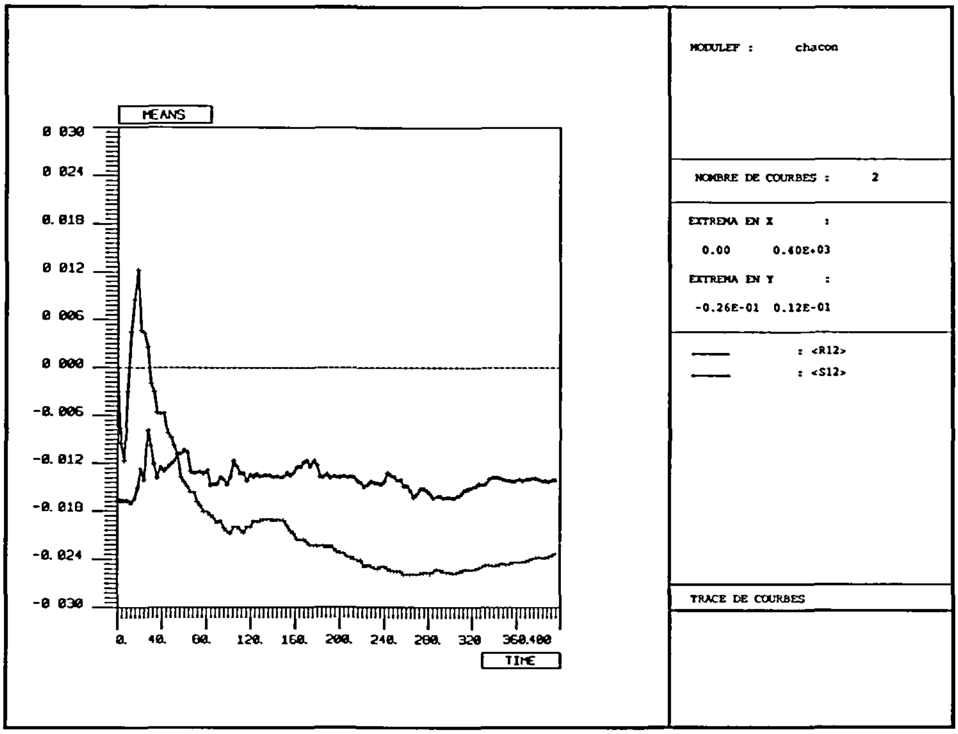

Figure 2 Time evolution of statistics r_{12} and s_{12}, with the same parameters as explained in *Figure 1*. A quasi-steady state is attempted by time $\tau\simeq 300$

Numerical results

Our computations have shown the existence of a quasy-steady statistical behaviour of the numerical solution provided by Algorithm 1, with initial conditions given by (34), when $N=8$ and $N=16$. This behaviour is reached for very long times ($\tau \simeq 400$) (see *Figures 1* to *5*). This, together with the high precision needed to solve problem (12)–(13) makes our code highly time-consuming. In practice, this introduces severe limitations in the size of N, that should not be greater than 16. However, some tests for intermediate times show that the statistics computed when $N=32$ and $N=16$ are very close. Furthermore, the steady states corresponding to $N=8$ and $N=16$ take similar values. Thus, taking N greater than 16 does not seem to be absolutely needed for our purposes of giving reasonably good computations of the closure terms for the turbulence model (4)–(8).

Let us remark that the theoretical symmetries of the problem are well reproduced by the numerical solution. *Figure 6* represents the time evolution of r_{13} and s_{13}, that should be zero from the theoretical symmetries. In our computations, these statistics are not exactly zero, although they take values about 100 times smaller than the other statistics. Note that this symmetry is suddenly broken at time $\tau \simeq 35$. One can prove that our algorithms are not stable in uniform norm for linear convection problems. In our case both algorithms are l^2-stable, but probably not uniformly stable. This may be the reason for such symmetry breaking.

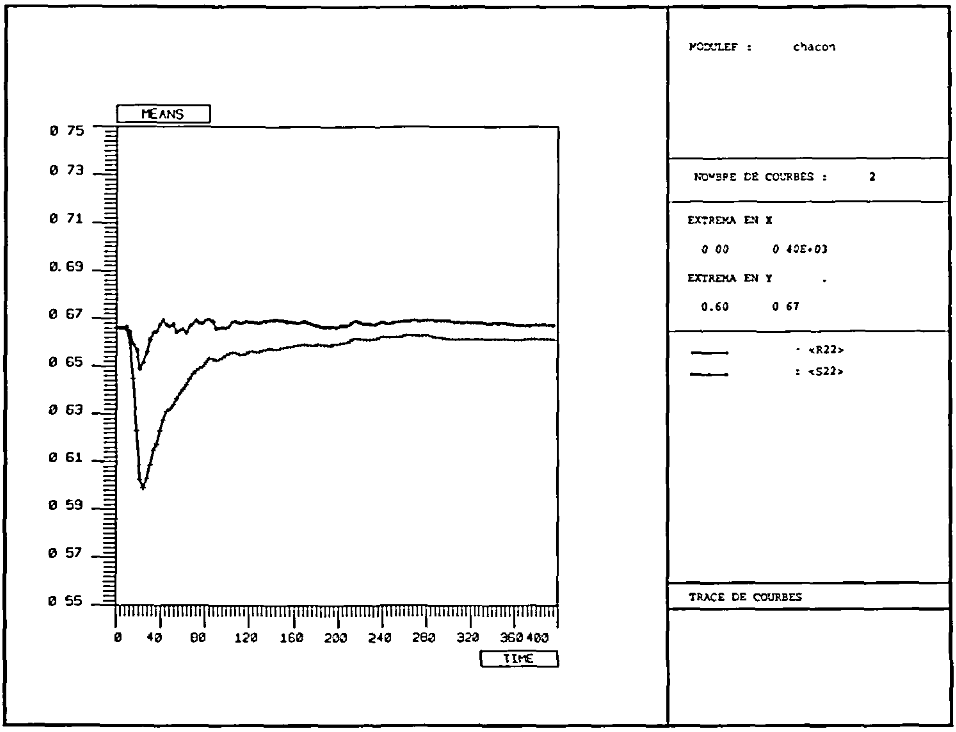

Figure 3 Time evolution of statistics r_{22} and s_{22}, with the same parameters as explained in *Figure 1*. Some slight oscillations are observed, although they are of small amplitude compared to that of those statistics

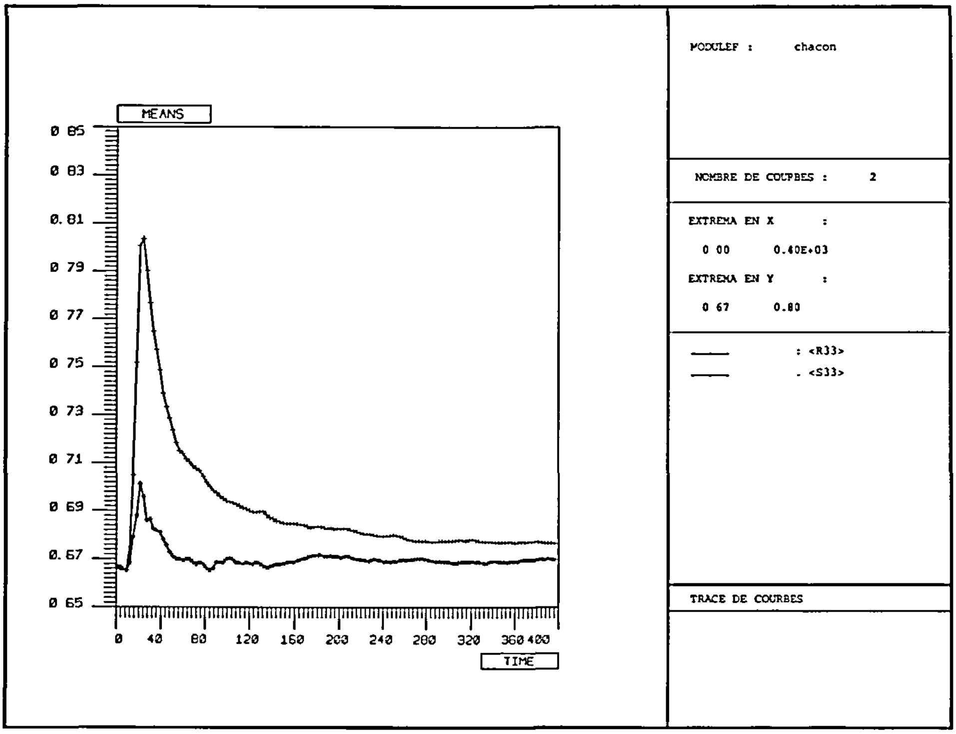

Figure 4 Time evolution of statistics r_{33} and s_{33}, with the same parameters as explained in *Figure 1*. A quasi-steady state is attempted by time $\tau \simeq 250$

A QUANTITATIVE TESTING OF COMPUTED CLOSURE TERMS

When the initial mean velocity field u_0 is constant, model (2) to (8) reduces to a classical k–ε model for globally isotropic turbulence[13,19]. Indeed, now the mean velocity field u is constant, and the equations for q and h can be combined to give an equation for the rate of viscous dissipation e of the mean kinetic energy,

$$e=\mu\langle|\nabla_y w|^2\rangle=\mu\langle|\nabla_y\times w|^2\rangle=\mu\frac{h^2}{q}\psi_q, \quad \text{with } \psi_q=\psi_q(I)$$

This model is the following:

$$\left.\begin{aligned} &q_{,t}+e=0\\ &e_{,t}+d\frac{e^2}{q}=0 \end{aligned}\right\} \tag{38}$$

where

$$d=2\frac{\psi_h}{\psi_q}-1 \tag{39}$$

is the dissipation coefficient of the rate of viscous dissipation e. In the classical theory of isotropic turbulence, the same model (38) describes globally isotropic turbulence[19]. The numerical constant

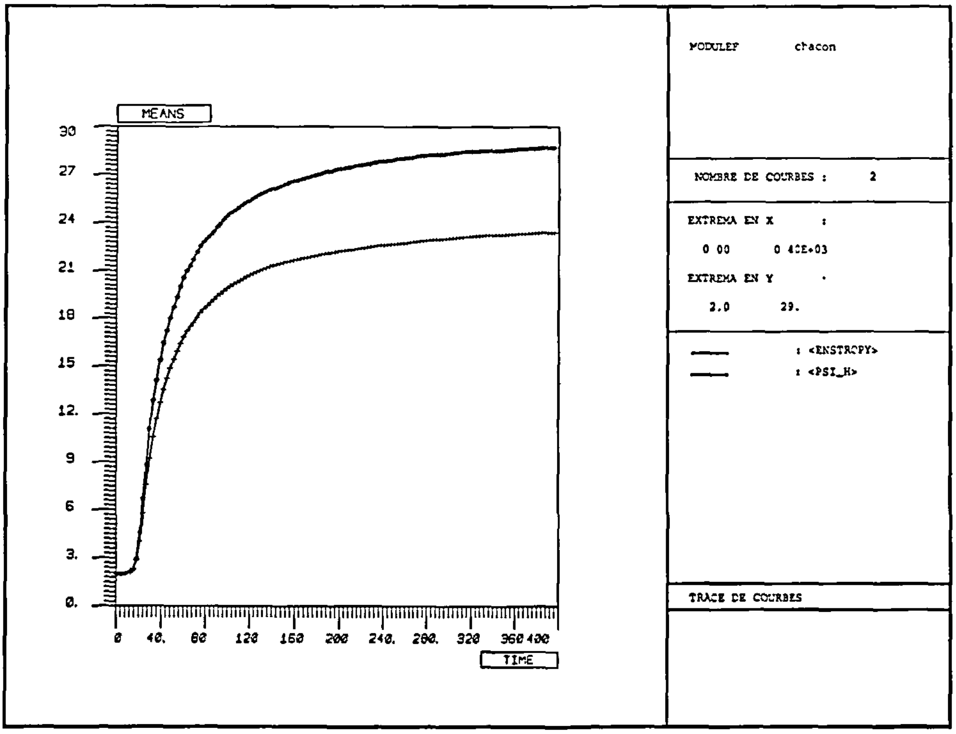

Figure 5 Time evolution of statistics ψ_q and ψ_h, with the same parameters as explained in *Figure 1*. A quasi-steady state is attempted by time $\tau \simeq 400$

d is obtained from experimental results and assumed to be universal in the sense that it is the same for similar experiments. Model (38) is physically meaningful as soon as $d>0$. Moreover, it is usually agreed that the value of d for fully developed turbulence is $d=1.92$. Here, we obtain d analytically from the universal microstructure problem (3) with $C=I$.

With an isotropic initial condition $\tilde{w}_N^0$, and $C=I$, Algorithm 1 yields $d\simeq 1.46$ for $N=8$, $d\simeq 1.54$ for $N=16$ and $d\simeq 1.52$ for $N=32$. This value is rather insensitive to the choice of the initial condition $\tilde{w}_N^0$, as one would expect from the universal character of the dissipation rate d. We think that this indicates that we are computing physically meaningful solutions of Euler equations, and that the closure terms that we obtain are relatively good approximations of the true values (up to an error of about 5% when $N=32$). A further improvement of the accuracy of these values would require a quite costly increasing of the number of degrees of freedom.

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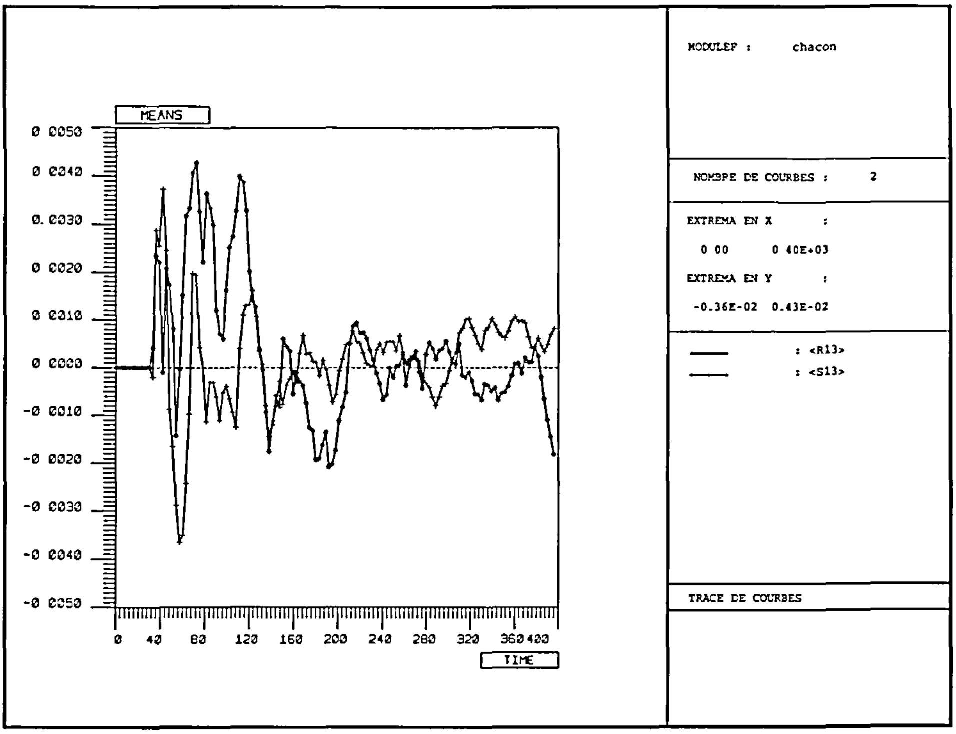

Figure 6 Time evolution of statistics r_{13} and s_{13}, with the same parameters as explained in *Figure 1*. Both statistics should be zero, if the theoretical symmetries given by (36) are to be respected. Note that both statistics oscillate around the value zero, and take absolute values about 100 times smaller than the other statistics

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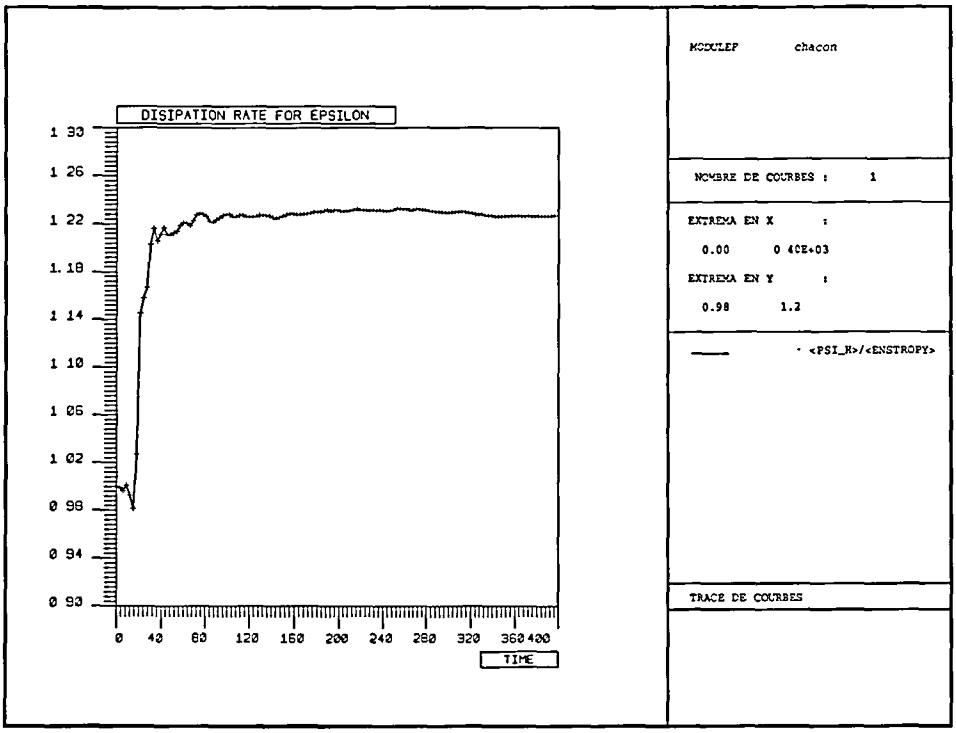

Figure 7 Time evolution of quotient ψ_h/ψ_q, that defines the dissipation rate d given by (39). A clear steady state is reached by time $\tau \simeq 100$

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